

MULTIFRACTAL ANALYSIS OF THE DIVERGENCE POINTS OF BIRKHOFF AVERAGES IN β -DYNAMICAL SYSTEMS

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ABSTRACT. This paper is aimed at a detailed study of the multifractal analysis of the so-called divergence points in the system of β -expansions. More precisely, let $([0, 1), T_\beta)$ be the β -dynamical system for a general $\beta > 1$ and $\psi : [0, 1] \mapsto \mathbb{R}$ be a continuous function. Denote by $A(\psi, x)$ all the accumulation points of $\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \psi(T^j x) : n \geq 1 \right\}$. The Hausdorff dimensions of the sets

$$\{x : A(\psi, x) \supset [a, b]\}, \quad \{x : A(\psi, x) = [a, b]\}, \quad \{x : A(\psi, x) \subset [a, b]\}$$

i.e., the points for which the Birkhoff averages of ψ do not exist but behave in a certain prescribed way, are determined completely for any continuous function ψ .

1. INTRODUCTION

Let (X, T) be a dynamical system. Given an integrable function ψ , call $x \in X$ a ψ -divergence point, or simply divergence point, if the limit of the Birkhoff averages

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(T^j x) := \lim_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) \quad (1.1)$$

does not exist. In the sense of Birkhoff's ergodic theorem, the divergence points are not detectable for any invariant probability measure. However, it is known that the divergence points can be large from the point of view of dimension theory, once ψ is not cohomologous to a constant (see for examples [2, 5, 23]). Moreover, as Ruelle said, points with converging Birkhoff averages can only see average behavior, while the divergence points would reflect a finer structure of the system [18]. This leads to a rich study on the structure of these points. Barreira and Schmeling initiated the study about the size of the divergence points in Markov systems [2], which was also extended to systems of conformal repeller, conformal horseshoes, β -expansions, see [3, 4, 20, 23] and references therein.

To have a better understanding of the divergence points and to provide extremely precise quantitative information about the distribution of the individual divergence points, Olsen [11] initiated a detailed study of the fractal structure of those points. More precisely, the multifractal decomposition

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sets were considered, where the Birkhoff averages diverge in a prescribed way:

- (1) $E_{\sup[a,b]} = \{x \in X : A(\psi, x) \supset [a, b]\};$
- (2) $E_{=[a,b]} = \{x \in X : A(\psi, x) = [a, b]\};$
- (3) $E_{\subset[a,b]} = \{x \in X : A(\psi, x) \subset [a, b]\},$

where $A(\psi, x)$ denotes the set of accumulation points of $\{\frac{1}{n}S_n\psi(x) : n \geq 1\}$. One is referred to a series of work of Olsen [9–12], Olsen & Winter [13, 14], Olsen, Baek & Snigireva [1] and references therein.

It should be pointed that most of them studied the dimensions of the sets defined above in the systems with Markov properties. In this paper, we focus on the β -expansions which is a non-Markov property for a general $\beta > 1$. This non-Markov property always plays a main barrier in studying the metrical properties of β -expansion. In order to understand better the non-Markov property and find ways to conquer difficulties caused by it, we aim at giving a detailed study of the multifractal analysis of the divergence points in β -expansions, by following the setting of Olsen given above.

Let us first fix some notation. Let $\beta > 1$ and T_β be the β -transformation given by

$$T_\beta(x) = \beta x - [\beta x], \quad x \in [0, 1)$$

where $[\cdot]$ denotes the integer part of a real number. It is well known that T_β is invariant and ergodic with respect to the Parry measure [15] given by $d\nu = \sum_{n: x \leq T_\beta^n 1} \beta^{-n} dx$. Then an application of the Birkhoff's ergodic Theorem yields that for any integrable function ψ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(T_\beta^j(x)) = \int \psi d\nu, \quad \nu\text{-almost surely.}$$

Recall that $A(\psi, x)$ is the set of accumulation points of the sequence $\{\frac{1}{n}S_n\psi(x)\}_{n \geq 1}$. Put $\mathfrak{Q}_\psi = \bigcup_{x \in [0, 1)} A(\psi, x)$. When ψ is continuous, \mathfrak{Q}_ψ is a closed interval and is given as

$$\mathfrak{Q}_\psi = \left\{ \int \psi d\mu : \mu \text{ is } T_\beta\text{-invariant} \right\}.$$

The classical multifractal analysis of Birkhoff averages in β -dynamical system was extensively studied by Fan, Feng and Wu [5] when β is a Parry number, and by Pfister and Sullivan [16] for general $\beta > 1$. For any $\alpha \in \mathfrak{Q}_\psi$, the dimension of the set

$$E_\alpha = \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) = \alpha \right\}$$

is given by a variational principle:

$$\dim_H E_\alpha = \sup \left\{ \frac{h_\mu}{\log \beta} : \mu \in \mathfrak{M}(T_\beta) \text{ and } \int \psi d\mu = \alpha \right\}, \quad (1.2)$$

where $\mathfrak{M}(T_\beta)$ is the collection of all T_β -invariant probability measures and h_μ is the measure theoretic entropy of μ (see [24], Chapter 4 for a definition of entropy). Denote $h_\beta(\alpha)$ the dimension of E_α for short.

In this paper, we focus on the dimensions of the sets $E_{\sup[a,b]}$, $E_{=[a,b]}$ and $E_{\subset[a,b]}$ defined before. Let ψ be a continuous function and a and b be two real numbers with $a < b$. Then $[a, b] \cap \mathfrak{L}_\psi$ is a closed interval, and we call it non-degenerate if it is nonempty. It is trivial to see that if $[a, b] \cap \mathfrak{L}_\psi$ is empty,

$$E_{\sup[a,b]} = [0, 1), \quad E_{=[a,b]} = \emptyset, \quad E_{\subset[a,b]} = \emptyset.$$

So we exclude this trivial case. Our main result is stated as follows.

Theorem 1.1. *Let $\beta > 1$ and ψ be a continuous function. Let a, b be two real numbers with $[a, b] \cap \mathfrak{L}_\psi$ non-degenerate. Then*

$$(1) \dim_{\text{H}} E_{\sup[a,b]} = \dim_{\text{H}} E_{=[a,b]} = \inf \{h_\beta(\alpha) : \alpha \in [a, b] \cap \mathfrak{L}_\psi\},$$

$$(2) \dim_{\text{H}} E_{\subset[a,b]} = \sup \{h_\beta(\alpha) : \alpha \in [a, b] \cap \mathfrak{L}_\psi\},$$

where $h_\beta(\alpha)$ is the dimension of E_α given in (1.2).

Let's give some words about the method used here compared with other related works.

- The setting here mostly follows that of Olsen's [11] (see also Olsen & Winter [10]), so we first compare our method with those introduced by them. The ideas to prove the first item in Theorem 1.1 are similar to them. But the difference is that we are in a non-finite Markov setting and some well known results in multifractal analysis cannot be applied directly. For the second item, they applied the large deviation theory, while a simple Lebesgue covering lemma is enough for us.
- Quite recently, B. Li and J. Li [8] considered the dimension of the set $E_{=[a,b]}$ when $\psi(x) = \omega_1(x, \beta)$, where $\omega_1(x, \beta)$ is the first digit of the β -expansion of x . More precisely, they considered the set

$$E_{=[a,b]}(\beta, \omega_1) := \left\{ x \in [0, 1) : A \left\{ \frac{1}{n} \sum_{j=1}^n \omega_j(x, \beta) \right\}_{n \geq 1} = [a, b] \right\}.$$

Their method is due to Schmeling [19], which is one very useful method in studying β -expansions. But it is not applicable here for a general function ψ . To make it clear, let us cite Schmeling's idea:

Let $\beta_0 < \beta$. One considers two systems $([0, 1), T_{\beta_0})$ and $([0, 1), T_\beta)$. Then define a map g between these two systems. More precisely, for any $x \in [0, 1)$, let

$$x = \frac{\omega_1(x, \beta_0)}{\beta_0} + \frac{\omega_2(x, \beta_0)}{\beta_0^2} + \dots + \frac{\omega_n(x, \beta_0)}{\beta_0^n} + \dots$$

be its β_0 -expansion. Then define

$$g(x) = \frac{\omega_1(x, \beta_0)}{\beta} + \frac{\omega_2(x, \beta_0)}{\beta^2} + \dots + \frac{\omega_n(x, \beta_0)}{\beta^n} + \dots$$

It was proved by Schmeling that g is $\frac{\log \beta_0}{\log \beta}$ -Hölder continuous. Thus if there is a set E' in $([0, 1), T_{\beta_0})$ such that $g(E')$ is a subset of E in $([0, 1), T_\beta)$, then it gives a lower bound of $\dim_H E$.

Now we turn back to the set $E_{=[a,b]}(\beta, \omega_1)$. It is clear that for each $n \geq 1$, $\omega_n(g(x), \beta) = \omega_n(x, \beta_0)$, so

$$g(E_{=[a,b]}(\beta_0, \omega_1)) \subset E_{=[a,b]}(\beta, \omega_1).$$

This enables them need only pay attention to the case when β is a Parry number. However, for a general function ψ , there is no clear relation between

$$\sum_{j=0}^{n-1} \psi(T_{\beta_0}^j(x)) \quad \text{and} \quad \sum_{j=0}^{n-1} \psi(T_\beta^j(g(x))).$$

So we have to find other way out. The method used in this paper is a successive approximation of β by Parry numbers, i.e. in different stage of the construction of a Moran subset of E , we use different Parry number to approximate β .

2. β -EXPANSIONS

In this section, we recall some basic properties of β -expansions. The β -expansion was first introduced by Rényi [17], which is given by the following algorithm. Let $\beta > 1$ and define

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \quad x \in [0, 1).$$

By taking

$$\omega_n(x, \beta) = \lfloor \beta T_\beta^{n-1} x \rfloor$$

recursively for each $n \geq 1$, every $x \in [0, 1)$ can be uniquely expanded into a finite or an infinite sequence

$$x = \frac{\omega_1(x, \beta)}{\beta} + \frac{\omega_2(x, \beta)}{\beta^2} + \cdots + \frac{\omega_n(x, \beta)}{\beta^n} + \cdots. \quad (2.1)$$

Call the series (2.1) the β -expansion of x and the sequence $\{\omega_n(x, \beta)\}_{n \geq 1}$ the digit sequence of x . We also write (2.1) as $x = (\omega_1(x, \beta), \cdots, \omega_n(x, \beta), \cdots)$.

A finite or an infinite sequence $(\omega_1, \omega_2, \cdots)$ is said to be *admissible* (with respect to the base β), if there exists an $x \in [0, 1)$ such that the digit sequence (in the β -expansion) of x begins with $\omega_1, \omega_2, \cdots$.

Denote by Σ_β^n the collection of all β -admissible sequences of length n and by Σ_β that of all infinite admissible sequences.

Now let us turn to the β -expansion of 1, which plays a crucial role in studying β -expansions. Let

$$1 = \frac{\omega_1(1, \beta)}{\beta} + \cdots + \frac{\omega_n(1, \beta)}{\beta^n} + \cdots,$$

be the β -expansion of 1. If it terminates, i.e. there exists $m \geq 1$ such that $\omega_m(1, \beta) \geq 1$ but $\omega_n(1, \beta) = 0$ for $n > m$ (those β are called Parry numbers), we put $(\omega_1^*(\beta), \omega_2^*(\beta), \cdots) = (\omega_1(1, \beta), \cdots, \omega_{m-1}(1, \beta), \omega_m(1, \beta) - 1)^\infty$, where

ω^∞ denotes the periodic sequence $(\omega, \omega, \omega, \dots)$. Otherwise, we put $(\omega_1^*(\beta), \omega_2^*(\beta), \dots)$ the infinite digit sequence $(\omega_1(1, \beta), \omega_2(1, \beta), \dots)$. In both cases, we call the sequence $(\omega_1^*(\beta), \omega_2^*(\beta), \dots)$ the infinite β -expansion of unity.

The lexicographical order $<_{\text{lex}}$ between two infinite sequences is defined as follows

$$(\omega_1, \omega_2, \dots, \omega_n, \dots) <_{\text{lex}} (\omega_1', \omega_2', \dots, \omega_n', \dots)$$

if there exists $k \geq 1$ such that $\omega_j = \omega_j'$ for $1 \leq j < k$, while $\omega_k < \omega_k'$.

The following result due to Parry [15] is a criterion for the admissibility of a sequence.

Theorem 2.1 ([15]). *(1) Let $\beta > 1$. A sequence of non-negative integers $\omega = (\omega_1, \omega_2, \dots)$ belongs to Σ_β if and only if*

$$\forall k \geq 1, (\omega_k, \omega_{k+1}, \dots) <_{\text{lex}} (\omega_1^*(\beta), \omega_2^*(\beta), \dots).$$

(2) The digit sequence $(\omega_1^(\beta), \omega_2^*(\beta), \dots)$ of the β -expansion of unity is monotone as a function of β . Therefore, if $1 < \beta_1 < \beta_2$,*

$$\Sigma_{\beta_1} \subset \Sigma_{\beta_2}, \Sigma_{\beta_1}^n \subset \Sigma_{\beta_2}^n \quad (\forall n \geq 1).$$

For any β -admissible sequence $(\omega_1, \dots, \omega_n) \in \Sigma_\beta^n$, the set

$$I_{n,\beta}(\omega_1, \dots, \omega_n) := \{x \in [0, 1) : \omega_j(x, \beta) = \omega_j, 1 \leq j \leq n\}$$

is called a cylinder of order n (with respect to the base β). From the algorithm of β -expansion, it is clear that the length of $I_{n,\beta}(\omega_1, \dots, \omega_n)$ satisfies that

$$|I_{n,\beta}(\omega_1, \dots, \omega_n)| \leq \beta^{-n},$$

where $|\cdot|$ denotes the length of an interval. If there is an equality, we call $I_{n,\beta}(\omega_1, \dots, \omega_n)$ a *full cylinder*.

The following simple fact will be referred to frequently, so we state it as a lemma.

Lemma 2.2. *Assume that ψ is a continuous function. Let $\epsilon > 0$ and $N \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $x \in [0, 1)$,*

$$\left| \frac{1}{n+N} S_{n+N} \psi(x) - \frac{1}{n} S_n \psi(x) \right| < \epsilon.$$

3. RELATIONSHIP BETWEEN (Σ_β, T_β) AND ITS SUBSYSTEMS

In this section, we approximate β from below by a sequence of Parry numbers β_N , and then introduce a quantitative relationship between (Σ_β, T_β) and $(\Sigma_{\beta_N}, T_{\beta_N})$, which is essential in our proof of the main theorem.

3.1. Approximating β from below. Recall that the β -expansion of unity is denoted by $(\omega_1^*(\beta), \omega_2^*(\beta), \dots)$.

For each $N \in \mathbb{N}$ with $\omega_N^*(\beta) \geq 1$, define β_N to be the unique positive root of the equation

$$1 = \frac{\omega_1^*(\beta)}{\beta_N^1} + \frac{\omega_2^*(\beta)}{\beta_N^2} + \dots + \frac{\omega_N^*(\beta)}{\beta_N^N}. \quad (3.1)$$

It is clear that $\beta_N \leq \beta$. Thus for $n \geq 1$, $\Sigma_{\beta_N}^n \subset \Sigma_{\beta}^n$. Also, we have that $\beta_N \rightarrow \beta$ as $N \rightarrow \infty$.

Proposition 3.1 ([6]). *For any $(\omega_1, \dots, \omega_n) \in \Sigma_{\beta_N}^n$, $I_{n+N, \beta}(\omega_1, \dots, \omega_n, 0^N)$ is full, or equivalently, for any $v \in \Sigma_{\beta}$, the concatenation $(\omega, 0^N, v)$ is still β -admissible.*

3.2. Relationship between Σ_{β} and Σ_{β_N} . In this short subsection, we cite a quantitative relation between Σ_{β} and Σ_{β_N} , which is borrowed from Tan and Wang [21].

Recall that β_N is defined in (3.1). So the infinite β_N -expansion of unity is the periodic sequence

$$(\omega_1^*(\beta), \dots, \omega_{N-1}^*(\beta), \omega_N^*(\beta) - 1)^{\infty}.$$

Now we induce a β_N -admissible sequence from a β -admissible sequence.

Given a β -admissible block $\omega = (\omega_1, \dots, \omega_n)$ with length n , we obtain a β_N -admissible sequence $\bar{\omega}$ by changing the blocks $(\omega_1^*(\beta), \dots, \omega_N^*(\beta))$ in ω from the left to the right with non-overlaps to $(\omega_1^*(\beta), \dots, \omega_N^*(\beta) - 1)$. Denote the resulting sequence by $\bar{\omega}$.

Proposition 3.2. $\bar{\omega} \in \Sigma_{\beta_N}^n$.

Define the map $\pi_N : \Sigma_{\beta}^n \rightarrow \Sigma_{\beta_N}^n$ as $\pi_N(\omega) = \bar{\omega}$.

Proposition 3.3. *For any $\bar{\omega} \in \Sigma_{\beta_N}^n$,*

$$\sharp \pi_N^{-1}(\bar{\omega}) \leq 2^{\frac{n}{N}},$$

i.e., the number of the inverse of $\bar{\omega} \in \Sigma_{\beta_N}^n$ is at most $2^{\frac{n}{N}}$.

These two elementary propositions have been proved very useful in studying the dimensional theory in β -expansions. For example, by using them, it was proved that the pressure function is continuous with respect to the system [21] and that the spectrum of the level set of the classic Birkhoff averages can be achieved by an approximating method [22]. Moreover, Propositions 4.1, 4.2 and 4.3 below, which are crucial in our argument, are also direct consequences of these elementary observations.

4. DIMENSIONAL NUMBER

In this section, we define several quantities which are closely related to the dimension of the sets in question.

From now on, we fix $\beta > 1$. All the cylinders in the sequel are cylinders with respect to the base β , so we write $I_n(\omega)$ for $I_{n,\beta}(\omega)$. Also we write $A(x)$ for $A(\psi, x)$.

Let $\alpha \in \mathfrak{L}_\psi$. For any $\epsilon > 0$ and $N \in \mathbb{N}$, define

$$\mathbb{F}(n, \alpha, \epsilon) = \left\{ \nu \in \Sigma_\beta^n : \left| \frac{1}{n} S_n \psi(x) - \alpha \right| < \epsilon, \text{ for some } x \in I_n(\nu) \right\},$$

$$\mathbb{F}_N(n, \alpha, \epsilon) = \left\{ \nu \in \Sigma_{\beta^N}^n : \left| \frac{1}{n} S_n \psi(x) - \alpha \right| < \epsilon, \text{ for some } x \in I_n(\nu) \right\}.$$

When we come to construct the desired Cantor set later, we will use $\mathbb{F}_N(\cdot)$ instead of $\mathbb{F}(\cdot)$ to avoid the barriers caused by the fact that the concatenation of two β -admissible words may not be admissible (Recall Proposition 3.1). The following proposition, which is a direct consequence of Proposition 3.3, says that we will not lose much if we do so.

Proposition 4.1 ([22]). *For any $\alpha \in \mathfrak{L}_\psi$,*

$$\#\mathbb{F}_N(n, \alpha, \epsilon) \leq \#\mathbb{F}(n, \alpha, \epsilon) \leq 2^{n/N} \#\mathbb{F}_N(n, \alpha, 2\epsilon).$$

As a consequence, the dimensional number $h_\beta(\alpha)$ of the set E_α can also be given as

$$h_\beta(\alpha) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log \#\mathbb{F}(n, \alpha, \epsilon)}{n \log \beta} = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \#\mathbb{F}_N(n, \alpha, \epsilon)}{n \log \beta}.$$

We restate the above proposition in another way, which will be frequently referred to.

Proposition 4.2. *For any $\delta > 0$, one can choose $\epsilon = \epsilon(\delta) > 0$, an integer $N = N(\delta, \epsilon) \in \mathbb{N}$ with $\omega_N^*(\beta) \geq 1$, and then $n \gg N$ correspondingly such that*

$$\left| \frac{\log \#\mathbb{F}_N(n, \alpha, \epsilon)}{n \log \beta} - h_\beta(\alpha) \right| < \delta. \quad (4.1)$$

The concavity and continuity of $h_\beta(\alpha)$ were proved in Propositions 4.3 and 4.4 of [22].

Proposition 4.3 ([22]). *$h_\beta(\alpha)$ is concave and continuous on \mathfrak{L}_ψ .*

5. PRELIMINARY RESULTS

In this section, we prove some preliminary results. Recall that

$$E_\alpha = \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) = \alpha \right\}$$

and $\dim_{\text{H}} E_\alpha = h_\beta(\alpha)$.

Proposition 5.1. *Let $\alpha \in \mathfrak{L}_\psi$. Define $E_{\ni \alpha} = \{x \in [0, 1) : A(x) \ni \alpha\}$. Then*

$$\dim_{\text{H}} E_\alpha = \dim_{\text{H}} E_{\ni \alpha}.$$

Proof. It's obvious that $E_\alpha \subset E_{\ni\alpha}$, so

$$\dim_{\text{H}} E_\alpha \leq \dim_{\text{H}} E_{\ni\alpha}.$$

Now we turn to the converse inequality. This is given by showing that

$$\dim_{\text{H}} E_{\ni\alpha} \leq h_\beta(\alpha).$$

Fix $\delta > 0$. By proposition 4.2, there exist $\epsilon = \epsilon(\delta)$ and $N(\delta, \epsilon) \in \mathbb{N}$ such that for any $n \geq N(\delta, \epsilon)$,

$$\log \#F(n, \alpha, \epsilon) \leq n(h_\beta(\alpha) + \delta) \log \beta. \quad (5.1)$$

On the other hand, it is clear that

$$\begin{aligned} E_{\ni\alpha} &\subset \left\{ x \in [0, 1) : \left| \frac{1}{n} S_n \psi(x) - \alpha \right| < \epsilon, \text{ i.o. } n \in \mathbb{N} \right\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{v \in F(n, \alpha, \epsilon)} I_n(v), \end{aligned}$$

where *i.o.* stands for *infinitely often*. Therefore, the s -dimensional Hausdorff measure of $E_{\ni\alpha}$ can be estimated as

$$\begin{aligned} \mathcal{H}^s(E_{\ni\alpha}) &\leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{v \in F(n, \alpha, \epsilon)} |I_n(v)|^s \\ &\leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \#F(n, \alpha, \epsilon) \cdot \beta^{-ns}, \end{aligned}$$

which is finite for any $s > h_\beta(\alpha) + \delta$ by (5.1). The arbitrariness of s yields that

$$\dim_{\text{H}} E_{\ni\alpha} \leq h_\beta(\alpha) + \delta.$$

□

The following is an elementary result, which shows that $A(x)$ is a closed interval.

Proposition 5.2. *$A(x)$ is a closed interval, more precisely,*

$$A(x) = \left[\liminf_{n \rightarrow \infty} \frac{S_n \psi(x)}{n}, \limsup_{n \rightarrow \infty} \frac{S_n \psi(x)}{n} \right]. \quad (5.2)$$

Proof. It suffices to show the inclusion “ \supset ” in (5.2). Fix a real number t with

$$\liminf_{n \rightarrow \infty} \frac{S_n \psi(x)}{n} < t < \limsup_{n \rightarrow \infty} \frac{S_n \psi(x)}{n}.$$

Then one has

$$\frac{1}{n} S_n \psi(x) < t, \text{ i.o. } n \in \mathbb{N}, \quad \text{and} \quad \frac{1}{n} S_n \psi(x) > t, \text{ i.o. } n \in \mathbb{N}.$$

This enables us to find a sequence $\{n_k\}_{k \geq 1}$ such that

$$\left(\frac{1}{n_k} S_{n_k} \psi(x) - t \right) \left(\frac{1}{n_k + 1} S_{n_k + 1} \psi(x) - t \right) \leq 0.$$

Thus,

$$\left| \frac{1}{n_k} S_{n_k} \psi(x) - t \right| \leq \left| \frac{1}{n_k} S_{n_k} \psi(x) - \frac{1}{n_k + 1} S_{n_k + 1} \psi(x) \right|.$$

By the boundedness of ψ , the right term turns to 0 as $k \rightarrow \infty$. This gives that

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} S_{n_k} \psi(x) = t.$$

□

The following dimension result about homogeneous Cantor set is a classic tool to estimate the Hausdorff dimension of a fractal set from below.

Let $\{m_i\}_{i \in \mathbb{N}}$ be a sequence of positive integers and $\{c_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers satisfying $m_i \geq 2$, $0 < c_i < 1$, $m_1 c_1 \leq \delta$ and $m_i c_i \leq 1$ ($i \geq 2$), where δ is some positive number. Let

$$D = \bigcup_{i \geq 0} D_i, \quad D_0 = \emptyset, \quad D_i = \{(\sigma_1, \dots, \sigma_i) : 1 \leq \sigma_j \leq m_j, 1 \leq j \leq i\}.$$

If $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$, $\tau = (\tau_1, \dots, \tau_m) \in D_m$, the concatenation of σ and τ is denoted by

$$\sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m).$$

Definition 5.3 ([7]). *Let (X, d) be a metric space. Suppose that $J \subset X$ is a closed subset with diameter $\delta > 0$. Let $\mathfrak{F} = \{J_\sigma : \sigma \in D\}$ be a collection of closed subsets of J with the properties*

- (1) $J_0 = J$;
- (2) *For any $i \geq 1$ and $\sigma \in D_{i-1}$, $J_{\sigma*1}, J_{\sigma*2}, \dots, J_{\sigma*m_i}$ are subsets of J_σ and $\text{int}(J_{\sigma*i}) \cap \text{int}(J_{\sigma*j}) = \emptyset$ ($i \neq j$), where $\text{int}(\cdot)$ denotes the interior of a set;*
- (3) *For any $i \geq 1$ and $\sigma \in D_{i-1}$, $1 \leq \ell \leq m_i$,*

$$\frac{|J_{\sigma*\ell}|}{|J_\sigma|} = c_i,$$

where $|\cdot|$ denotes the diameter. Then

$$\mathbb{C}_\infty := \bigcap_{i \geq 1} \bigcup_{\sigma \in D_i} J_\sigma$$

is called a homogeneous Cantor set determined by \mathfrak{F} . For each $i \geq 1$, we call the union $\mathbb{C}_i := \bigcup_{\sigma \in D_i} J_\sigma$ the i th generation of \mathbb{C}_∞ .

Lemma 5.4 ([7]). *For the homogeneous Cantor set defined above, we have*

$$\dim_{\text{H}} \mathbb{C}_\infty \geq \liminf_{i \rightarrow \infty} \frac{\log m_1 \cdots m_i}{-\log c_1 \cdots c_{i+1} m_{i+1}}.$$

6. PROOF OF THEOREM 1.1

Recall that \mathfrak{L}_ψ is a closed interval. So without loss of generality, we assume that $[a, b] \subset \mathfrak{L}_\psi$.

6.1. The first item in Theorem 1.1.

On the one hand, by the concavity of $h_\beta(\cdot)$, one has

$$\min \{h_\beta(\alpha) : \alpha \in [a, b]\} = \min \{h_\beta(a), h_\beta(b)\} := s_*.$$

On the other hand, it is obvious that $E_{=[a,b]} \subset E_{\supset[a,b]}$. So to get the desired result, it suffices to show that

$$s_* \leq \dim_{\text{H}} E_{=[a,b]}, \quad \dim_{\text{H}} E_{\supset[a,b]} \leq s_*.$$

The second inequality is a direct corollary from Proposition 5.1, so we need only pay attention to the first one.

Since $A(x)$ is a closed interval (Proposition 5.2), we can rewrite $E_{=[a,b]}$ as

$$E_{=[a,b]} = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) = a, \limsup_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) = b \right\}.$$

We will construct a homogeneous Cantor set $\mathbb{C}_\infty \subset E_{=[a,b]}$ with Hausdorff dimension bounded from below by s_* .

We fix some notations.

- At first, fix $\delta > 0$. By Proposition 4.2, we can choose a sequence of triples $(\epsilon_k, N_k, n_k)_{k \geq 1}$ (recursively) such that $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, $N_k > N_{k-1}$, $n_k/(n_k + N_k) \geq 1 - \delta$ and for each $k \geq 1$,

$$\left| \frac{\log \# \mathbb{F}_{N_{2k-1}}(n_{2k-1}, a, \epsilon_{2k-1})}{n_{2k-1} \log \beta} - h_\beta(a) \right| < \delta \quad (6.1)$$

$$\left| \frac{\log \# \mathbb{F}_{N_{2k}}(n_{2k}, b, \epsilon_{2k})}{n_{2k} \log \beta} - h_\beta(b) \right| < \delta. \quad (6.2)$$

- Secondly, we choose a sequence of integers $\{\ell_k\}_{k \geq 1}$ such that for each $k \geq 1$,

$$\ell_k \gg n_{k+1} + N_{k+1}, \quad \ell_k \gg \ell_1(n_1 + N_1) + \cdots + \ell_{k-1}(n_{k-1} + N_{k-1}) \quad (6.3)$$

and let $n_0 = N_0 = l_0 = 0$.

- Thirdly, we define

$$\mathbb{D}_k = \begin{cases} \left\{ \omega = (v, 0^{N_k}) : v \in \mathbb{F}_{N_k}(n_k, a, \epsilon_k) \right\}, & \text{when } k \text{ is odd;} \\ \left\{ \omega = (v, 0^{N_k}) : v \in \mathbb{F}_{N_k}(n_k, b, \epsilon_k) \right\}, & \text{when } k \text{ is even.} \end{cases}$$

- At last, we give an estimation. If n_k is chosen sufficiently large compared with N_k , then by Lemma 2.2, for each $\omega \in \mathbb{D}_k$ and every $x \in I_{n_k+N_k}(\omega)$, we have

$$\left| \frac{1}{n_k + N_k} \sum_{j=0}^{n_k+N_k-1} \psi(T_\beta^j x) - a \text{ (or } b) \right| < 2\epsilon_k, \quad (6.4)$$

according as k is odd or even.

Remark 1. Note that the words v in $\mathbb{F}_{N_k}(n_k, *, \epsilon_k)$ are β_{N_k} -admissible. So the extra term 0^{N_k} ensures that every word ω in \mathbb{D}_k can concatenate any other β -admissible words freely by Proposition 3.1.

Now we are in the position to construct a subset \mathbb{C}_∞ of $E_{=[a,b]}$ generation by generation. Firstly, let $\mathbb{C}_0 = [0, 1]$.

The generations $\{\mathbb{C}_i\}_{1 \leq i \leq \ell_1}$. For each $1 \leq i \leq \ell_1$, set

$$\mathbb{C}_i = \bigcup_{\omega_1 \in \mathbb{D}_1, \dots, \omega_i \in \mathbb{D}_1} I_{i(n_1+N_1)}(\omega_1, \dots, \omega_i).$$

By the remark given above and Proposition 3.1, all the cylinders in \mathbb{C}_i are full cylinders of order $i(n_1+N_1)$. Then the ratio of the diameter of a cylinder in \mathbb{C}_{i-1} with that of a cylinder in \mathbb{C}_i is $\beta^{-(n_1+N_1)}$. Moreover, it is also clear that each cylinder in \mathbb{C}_{i-1} contains $\#\mathbb{D}_1$ elements in \mathbb{C}_i . So, by borrowing the notation from Definition 5.3 of a homogeneous Cantor set, we have

$$c_i = \beta^{-(n_1+N_1)}, \quad m_i = \#\mathbb{D}_1 = \#\mathbb{F}_{N_1}(n_1, a, \epsilon_1), \quad \text{for } 1 \leq i \leq \ell_1. \quad (6.5)$$

The inductive step. Assume that the first $(\ell_1 + \dots + \ell_{k-1})$ th generations have been well defined. Note that $\mathbb{C}_{\ell_1+\dots+\ell_{k-1}}$ consists of a collection of full cylinders of order

$$\ell_1(n_1 + N_1) + \dots + \ell_{k-1}(n_{k-1} + N_{k-1}) := t_{k-1}.$$

For notational simplification, we write a general element in $\mathbb{C}_{\ell_1+\dots+\ell_{k-1}}$ as $I_{t_{k-1}}(\omega^{(k-1)})$. Note that here $\omega^{(k-1)}$ is a word of length t_{k-1} which can concatenate any other β -admissible words (by Remark 1).

Now we define the generations

$$\{\mathbb{C}_i : \ell_1 + \dots + \ell_{k-1} < i \leq \ell_1 + \dots + \ell_{k-1} + \ell_k\}.$$

Write $j = i - (\ell_1 + \dots + \ell_{k-1})$. Then set

$$\mathbb{C}_i = \bigcup_{I_{t_{k-1}}(\omega^{(k-1)}) \subset \mathbb{C}_{\ell_1+\dots+\ell_{k-1}}} \bigcup_{\omega_1 \in \mathbb{D}_k, \dots, \omega_j \in \mathbb{D}_k} I_{t_{k-1}+i(n_k+N_k)}(\omega^{(k-1)}, \omega_1, \dots, \omega_j).$$

Similar to (6.5), we have, for $\ell_1 + \dots + \ell_{k-1} < i \leq \ell_1 + \dots + \ell_{k-1} + \ell_k$,

$$c_i = \beta^{-(n_k+N_k)}, \quad m_i = \#\mathbb{D}_k = \#\mathbb{F}_{N_k}(n_k, a, \epsilon_k) \quad \text{or} \quad \#\mathbb{F}_{N_k}(n_k, b, \epsilon_k) \quad (6.6)$$

according as k is odd or even.

The desired homogeneous Cantor set is defined as

$$\mathbb{C}_\infty = \bigcap_{i=0}^{\infty} \mathbb{C}_i.$$

We claim that

Proposition 6.1. $\mathbb{C}_\infty \subset E_{=[a,b]}$.

Proof. The proof is done by some elementary estimations, so we only check that for each $x \in \mathbb{C}_\infty$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) \leq b \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{1}{t_{2k}} S_{t_{2k}} \psi(x) \geq b. \quad (6.7)$$

Bear in mind the construction of \mathbb{C}_∞ , the formula (6.4) and

$$t_k = \ell_1(n_1 + N_1) + \dots + \ell_k(n_k + N_k).$$

(i). For each $n \gg 1$, let k and then $0 \leq \ell < \ell_k$, $0 \leq j < n_k + N_k$ be the integers such that

$$t_{k-1} \leq n = t_{k-1} + \ell(n_k + N_k) + j < t_k.$$

Since $a \leq b$, then by (6.4), we have

$$\begin{aligned} S_n \psi(x) &\leq \ell_1 \cdot (n_1 + N_1)(b + 2\epsilon_1) + \cdots + \ell_{k-1} \cdot (n_{k-1} + N_{k-1})(b + 2\epsilon_{k-1}) \\ &\quad + \ell \cdot (n_k + N_k)(b + 2\epsilon_k) + j \|\psi\|_\infty. \end{aligned}$$

Recall the choice of ℓ_k (see the first formula in (6.3)) and the fact that $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. One can say that

$$n_i + N_i = n_i + o(n_i), \quad j \|\psi\|_\infty = o(\ell_{k-1}) = o(t_{k-1}).$$

Thus it follows that

$$S_n \psi(x) \leq b(t_{k-1} + \ell(n_k + N_k)) + o(t_{k-1}) \leq b \cdot n + o(n).$$

Therefore, the first assertion in (6.7) follows.

(ii). By the second formula in (6.3) and the inequality (6.4), we have

$$S_{t_{2k}} \psi(x) \geq o(\ell_{2k}) + \ell_{2k}(n_{2k} + N_{2k})(b - 2\epsilon_{2k}).$$

Using the second formula in (6.3) again, which enables us to say that $\ell_{2k}(n_{2k} + N_{2k}) = t_{2k} + o(t_{2k})$, the second assertion in (6.7) thus follows. \square

6.2. Hausdorff dimension of \mathbb{C}_∞ . We apply Lemma 5.4 to give the lower bound of $\dim_{\text{H}} \mathbb{C}_\infty$.

Now we collect the information about c_i and m_i : for $\ell_1 + \cdots + \ell_{k-1} < i \leq \ell_1 + \cdots + \ell_{k-1} + \ell_k$,

$$\begin{aligned} c_i &= \beta^{-(n_k + N_k)}, \\ \beta^{n_k(h_\beta(a) - \delta)} &\leq m_i \leq \beta^{n_k(h_\beta(a) + \delta)} \quad \text{or} \quad \beta^{n_k(h_\beta(b) - \delta)} \leq m_i \leq \beta^{n_k(h_\beta(b) + \delta)}, \end{aligned}$$

according as k is odd or even (see the formulae (6.6), (6.1) and (6.2)).

Since $\ell_{k-1} \gg n_k + N_k$, the term $\log(c_{i+1} \cdot m_{i+1})$ is negligible compared with $\log(c_1 \cdots c_i)$. So, by Lemma 5.4,

$$\begin{aligned} \dim_{\text{H}} \mathbb{C}_\infty &\geq \liminf_{i \rightarrow \infty} \frac{\log(m_1 \cdots m_i)}{-\log(c_1 \cdots c_i)} \geq \inf_{i \geq 1} \frac{\log m_i}{-\log c_i} \\ &\geq (1 - \delta) \min \{h_\beta(a) - \delta, h_\beta(b) - \delta\}. \end{aligned}$$

Then by the arbitrariness of $\delta > 0$, it follows that

$$\dim_{\text{H}} E_{=[a,b]} \geq \min \{h_\beta(a), h_\beta(b)\}.$$

6.3. **The second item in Theorem 1.1.** Recall that

$$E_{\subset[a,b]} = \{x : A(x) \subset [a, b]\}.$$

Write $s^* = \sup \{h_\beta(\alpha) : \alpha \in [a, b]\}$.

Lower bound. It is clear that for each $\alpha \in [a, b]$, $E_\alpha \subset E_{\subset[a,b]}$. Thus

$$\dim_H E_{\subset[a,b]} \geq h_\beta(\alpha), \text{ for all } \alpha \in [a, b].$$

Upper bound. To get the upper bound, we first cite a Lebesgue covering Lemma from [24]:

Lemma 6.2 ([24]). *Let (X, d) be a compact metric space and \mathfrak{D} an open cover of X . Then there exists a $\delta > 0$ such that each subset of X of diameter less than or equal to δ lies in some member of \mathfrak{D} . (Such a δ is called a Lebesgue number of \mathfrak{D} .)*

Fix $\eta > 0$. Recall the definition of $h_\beta(\alpha)$ (Proposition 4.1). Then for any $\alpha \in [a, b]$, there exist $\epsilon_\alpha = \epsilon(\alpha, \eta)$ and $N(\eta, \epsilon_\alpha) \in \mathbb{N}$ such that for any $n \geq N(\eta, \epsilon_\alpha)$, we get

$$\log \#F(n, \alpha, \epsilon_\alpha) \leq n(h_\beta(\alpha) + \eta) \log \beta.$$

So for each $\alpha \in [a, b]$, we have

$$\log \#F(n, \alpha, \epsilon_\alpha) \leq n(s^* + \eta) \log \beta. \quad (6.8)$$

Clearly, $\{B(\alpha, \epsilon_\alpha) : \alpha \in [a, b]\}$ is an open covering of the closed interval $[a, b]$. Then by Lemma 6.2, let δ be a lebesgue number of this cover. Take $\{[a_k, b_k] : k \geq 1\}$ a collection of closed intervals with diameter at most $\frac{\delta}{4}$ such that

$$[a, b] = \bigcup_{k \geq 1} [a_k, b_k].$$

Then

$$E_{\subset[a,b]} = \{x : A(x) \subset [a, b]\} \subset \bigcup_{k \geq 1} E^{(k)}$$

where $E^{(k)} = \{x : A(x) \cap [a_k, b_k] \neq \emptyset\}$.

For each $k \geq 1$, by using Lebesgue covering Lemma, there exists $\alpha \in [a, b]$ such that

$$[a_k - \delta/4, b_k + \delta/4] \subset B(\alpha, \epsilon_\alpha).$$

On the other hand, for each $x \in E^{(k)}$, there exists $t_x \in [a_k, b_k]$ such that $t_x \in A(x)$. Then there are infinitely many n such that

$$\left| \frac{1}{n} S_n \psi(x) - t_x \right| < \delta/4.$$

Since

$$B(t_x, \delta/4) \subset [a_k - \delta/4, b_k + \delta/4] \subset B(\alpha, \epsilon_\alpha),$$

we have

$$E^{(k)} \subset \left\{ x : \left| \frac{1}{n} S_n \psi(x) - \alpha \right| < \epsilon_\alpha \text{ i.o. } n \in \mathbb{N} \right\}.$$

Therefore,

$$E^{(k)} \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{v \in \mathbb{F}(n, \alpha, \epsilon_{\alpha})} I_n(v).$$

Then the s -dimensional Hausdorff measure of $E^{(k)}$ can be estimated as

$$\begin{aligned} \mathcal{H}^s(E^{(k)}) &\leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{v \in \mathbb{F}(n, \alpha, \epsilon_{\alpha})} |I_n(v)|^s \\ &\leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \#\mathbb{F}(n, \alpha, \epsilon_{\alpha}) \cdot \beta^{-ns}, \end{aligned}$$

which is finite for any $s > s^* + \eta$ by (6.8). The arbitrariness of s yields that

$$\dim_{\mathrm{H}} E^{(k)} \leq s^* + \eta.$$

Hence

$$\dim_{\mathrm{H}} E_{C[a,b]} \leq \sup_{k \geq 1} \dim_{\mathrm{H}} E^{(k)} \leq s^* + \eta.$$

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